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COMPUTATIONAL ESTIMATION OF THE ORDER OF $\zeta(\frac{1}{2}+it)$

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Abstract. The paper describes a search for increasingly large extrema (ILE) of $|\zeta(\frac{1}{2}+it)|$ in the range $0 \le t \le 10^{13}$. For $t \le 10^6$, the complete set of ILE (57 of them) was determined. In total, 162 ILE were found, and they suggest that $\zeta(\frac{1}{2}+it) = \Omega(t^{2/\sqrt{\log t \log\log t}})$. There are several regular patterns in the location of ILE, and arguments for these regularities are presented. The paper concludes with a discussion of prospects for further computational progress.

1. Introduction

Riemann's zeta function on the critical line, $\zeta(\frac{1}{2}+it)$, is unbounded. Balasubramanian and Ramachandra have shown in 1977 [1] that

$$\zeta(\tfrac{1}{2}+it) = \Omega(t^{\frac{3}{4\sqrt{\log t \log\log t}}})$$

whereas Huxley proved in 1993 [3] that

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{89}{570}+\varepsilon}) \quad \text{for every } \varepsilon > 0.$$

This leaves a considerable gap between the Ω- and O-results. Already in 1908, Lindelöf conjectured a much stronger O-bound [4]

$$\zeta(\tfrac{1}{2}+it) = O(t^{\varepsilon}) \quad \text{for every } \varepsilon > 0.$$

The truth of this conjecture, known as Lindelöf's hypothesis, would follow from that of Riemann's hypothesis, since the latter can only hold if [8]

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{C}{\log\log t}}) \quad \text{for some } C > 0.$$

Since $|\zeta(\frac{1}{2}+it)| = |Z(t)|$, where $Z(t)$ is the Riemann-Siegel Z function, the conjectures and results about the order of $\zeta(\frac{1}{2}+it)$ may, and henceforth, will be stated more compactly in terms of $Z(t)$. As $Z(t)$ is an even function, any discussion about its behavior will be restricted to $t \in \mathbb{R}_+$ without loss of generality, so "at values of t smaller than T" will always mean $0 \le t < T$. The acronym *ILE* will be used for *increasingly large extrema* of $|Z(t)|$, and an interval bounded by two consecutive zeros of $Z(t)$ will be referred to as an *interzero interval*.

A computational search for large values of $|Z(t)|$ obviously cannot provide rigorous Ω- and O-results. Still, the results presented in this paper show that with a sufficiently comprehensive set of ILE determined in a sufficiently large t-interval, certain regularities in the values of $Z(t)$ at ILE become detectable. The values of ILE in the interval $0 \le t \le 10^{13}$ suggest that the Ω-bound of $Z(t)$ could be

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improved substantially. On the other hand, a much broader t-interval would have to be investigated to suggest potential improvements of the O-bound of $Z(t)$.

2. Methods of computation

2.1. **General.** The computations were performed on a PC equipped with a 1700 MHz Intel Pentium 4 processor. The values of $Z(t)$ and $\vartheta(t)$ were computed with Mathematica 4.0 (Wolfram Research, Urbana, IL, USA) using the `RiemannSiegelZ` and `RiemannSiegelTheta` routines, respectively. The search algorithm was run using 16-digit precision, while the values of ILE were determined with 24-digit precision. Least-squares regression was performed with Sigma Plot 6.0 (SPSS Science, Chicago, IL, USA).

2.2. **Determination of all ILE for $0 \leq t \leq 10^6$.** In $\mathcal{T}_1 := [0, 10^6]$, $Z(t)$ has 1747146 zeros,[1] and Riemann's hypothesis is never violated there [7]. Hence $Z(t)$ has exactly one local extremum in each interzero interval in $\mathcal{T}_1$ [2]. Together with three extrema below the first zero, there are thus 1747148 local extrema of $Z(t)$ in $\mathcal{T}_1$. Of these extrema, 57 are ILE, forming the list $\mathcal{Z}_1$ (see the Appendix).

Section 4.1 presents two plausible theoretical arguments for the proximity of large extrema of $|Z(t)|$ to the points $t_k := \frac{2k\pi}{\log 2}$, $k \in \mathbb{N}$. This indeed appears to be the case — *each* of the interzero intervals containing an ILE of $\mathcal{Z}_1$ also contains such a point. Furthermore, in all cases, $|Z(t_k)|$ exceeds 47% of the maximum value of $|Z(t)|$ in the same interzero interval, and on average, it exceeds 91% of that value.

2.3. **Search for ILE for $10^6 < t \leq 10^9$.** The regularity in the location of ILE in $\mathcal{Z}_1$ suggests that many large $|Z(t)|$ are located in the interzero intervals containing a point t_k and a relatively large $|Z(t_k)|$. The search for ILE in $\mathcal{T}_2 := (10^6, 10^9]$ was performed as follows:

(1) $Z(t_k)$ was computed;
(2) if $|Z(t_k)|$ exceeded 20% of the largest ILE for smaller t, the local extremum was computed;
(3) if $|Z(t)|$ at the extremum exceeded the largest ILE for smaller t, it was added to the list $\mathcal{Z}_2$.

None of the ILE in $\mathcal{T}_1$ would have been missed by this algorithm. In total, the list $\mathcal{Z}_2$ consists of 43 extrema, and they are given in the Appendix.

Section 4.2 sketches an argument for another regular pattern in the location of large extrema of $|Z(t)|$. Denoting by d_p the absolute deviation of $\frac{k \log p}{\log 2}$ (p prime) from an integer, a large $|Z(t_k)|$ is likely if $d_3, d_5, d_7, \ldots$, are relatively small. The list $\mathcal{Z}_2$ provides a sample of d_p for 43 ILE in $\mathcal{T}_2$. The increase of d_p with p in $\mathcal{Z}_2$ is rather rapid; thus $\text{mean}(d_3) = 0.0281\ldots$, $\max(d_3) = 0.0861\ldots$, and $\text{mean}(d_{47}) = 0.1581\ldots$, $\max(d_{47}) = 0.4966\ldots$.

2.4. **Search for ILE for $10^9 < t \leq 10^{13}$.** Since in $\mathcal{Z}_2$ the d_p for small p are small, ILE near t_k with large d_p are unlikely. The ranges of permitted d_p were chosen on the basis of their respective values in $\mathcal{Z}_2$, and the search for ILE in $\mathcal{T}_3 := (10^9, 10^{13}]$ was performed as follows:

(1) the values of d_p, $3 \leq p \leq 17$, were checked to be within prescribed ranges: $d_3 \leq 0.10$, $d_5 \leq 0.15$, $d_7 \leq 0.20$, $d_{11} \leq 0.25$, $d_{13} \leq 0.28$, $d_{17} \leq 0.30$;

[1] The list of zeros, accurate to $\pm 10^{-9}$, was kindly provided by Dr. Andrew M. Odlyzko.

(2) if the value of k qualified, $Z(t_k)$ was computed;
(3) if $|Z(t_k)|$ exceeded 20% of the largest ILE for smaller t, the local extremum was computed;
(4) if $|Z(t)|$ at the extremum exceeded the largest ILE for smaller t, it was added to the list $\mathcal{Z}_3$.

None of the ILE found in $\mathcal{T}_2$ would have been missed with this choice of bounds on $d_3, \ldots, d_{17}$. In total, the list $\mathcal{Z}_3$ consists of 62 extrema, and they are given in the Appendix.

3. Results and discussion

Let

$$a(t) := \frac{\log|Z(t)|}{\log t} \qquad \text{and} \qquad b(t) := \frac{\log|Z(t)|\,\sqrt{\log\log t}}{\sqrt{\log t}}.$$

Denoting $\limsup_{t\to\infty} a(t) = A$ and $\limsup_{t\to\infty} b(t) = B$, we have $0 \le A \le \frac{89}{570}$ by the theorem of Huxley, and $\frac{3}{4} \le B \le \infty$ by the theorem of Balasubramanian and Ramachandra. At sufficiently large t, where large $|Z(t)|$ start to reflect the actual order of $Z(t)$, the values of $a(t)$ and $b(t)$ at ILE should start to approach the true values of A and B, respectively.

Figure 1 shows the values of $a(t)$ and $b(t)$ for ILE in $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3$, excluding $|Z(0)|$, and for $\left|Z(4.257... \times 10^{15})\right| = 855.3...$ in the vicinity of a point located by Odlyzko [5]. The values of $a(t)$ at ILE seem to delineate a monotonically decreasing asymptote for $t > 10^3$, but these a-values are too large to suggest a stronger upper bound of A than the value $\frac{89}{570} = 0.1561...$ imposed by the theorem of Huxley. On the other hand, the values of $b(t)$ at ILE seem to delineate a monotonically increasing asymptote for all t, exceeding for $t > 10^2$ the lower bound of B imposed by the theorem of Balasubramanian and Ramachandra. Close to the upper bound of the investigated t-range, we have $b(t) > 2$, and the asymptotic increase of $b(t)$ seems to continue, which suggests that

$$Z(t) = \Omega(t^{2/\sqrt{\log t \, \log\log t}}).$$

It seems likely that the extension of the range of ILE to larger t would allow to strengthen this tentative estimate.

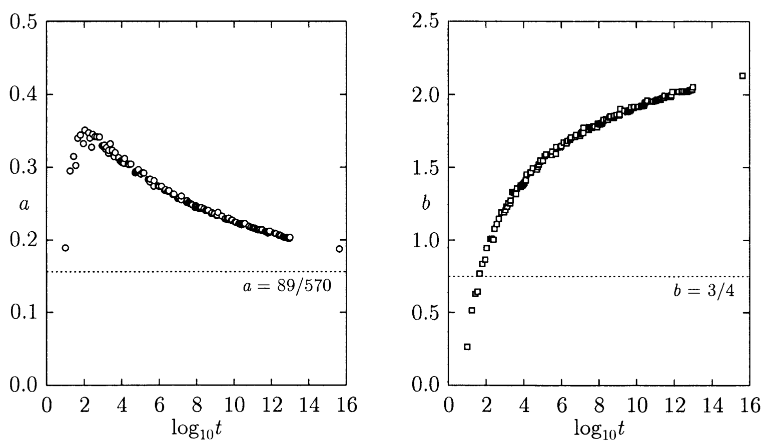

Figure 1.

4. Patterns in the location of large extrema

4.1. **Proximity of $\frac{t\log 2}{2\pi}$ to $\mathbb{N}$.** As described in Section 2.2, of the subzero interval and the 55 interzero intervals containing ILE in $\mathcal{Z}_1$, *each* also contains a point $t_k := \frac{2k\pi}{\log 2}$. Plausible arguments for this can be derived from at least two starting points.

Argument A. From the well-known formula

$$\zeta(\sigma+it) = \frac{1}{(1-2^{1-\sigma-it})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma+it}} \quad \text{for } \sigma>0$$

we have

$$|Z(t)| = (3-2\sqrt{2}\cos(t\log 2))^{-1/2}\left|\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1/2+it}}\right|.$$

Large $|Z(t_k)|$ can then be explained by periodicity of $(3-2\sqrt{2}\cos(t\log 2))^{-1/2}$, with maxima of $\sqrt{2}+1$ at $t=\frac{2k\pi}{\log 2}$ and minima of $\sqrt{2}-1$ at $t=\frac{(2k-1)\pi}{\log 2}$.

Argument B. We invoke the main sum in the Riemann-Siegel formula

$$Z_0(t) = 2\sum_{1\le n\le\sqrt{\frac{t}{2\pi}}}\frac{\cos(\vartheta(t)-t\log n)}{\sqrt{n}},$$

where $\vartheta(t)$ is the Riemann-Siegel theta function. At the points $t=t_k$ we have $\cos(\vartheta(t_k)-t_k\log n) = \cos\vartheta(t_k)$ for summands with $n=2^m$, $m\in\{0\}\cup\mathbb{N}$, which therefore reinforce each other (i.e., have the same sign), and

$$Z_0(t_k) = 2\cos\vartheta(t_k)\sum_{\substack{1\le n\le\sqrt{\frac{t}{2\pi}},\\ n=2^m}}\frac{1}{\sqrt{n}} \;+\; 2\sum_{\substack{3\le n\le\sqrt{\frac{t}{2\pi}},\\ n\ne 2^m}}\frac{\cos(\vartheta(t_k)-t_k\log n)}{\sqrt{n}}.$$

4.2. **Proximity of $k\frac{\log p}{\log 2}$ to $\mathbb{N}$.** For $\{t_{k(3)}\}\subset\{t_k\}$, for which $\frac{k\log 3}{\log 2}\approx l\in\mathbb{N}$, we have $\frac{2k\pi}{\log 2}\approx\frac{2l\pi}{\log 3}$, so $\cos(\vartheta(t_k)-t_k\log n)\approx\cos\vartheta(t_k)$ for summands with $n=3^m$ and $n=2^m3^{m'}$, with $m,m'\in\mathbb{N}$, and these summands are also mutually reinforcing. Denoting $\{t_{k(3,5)}\}\subset\{t_{k(3)}\}$, for which $\frac{k\log 5}{\log 2}$ is also close to an integer, mutual reinforcement also occurs for summands with $n=5^m$, $n=2^m5^{m'}$, $n=3^m5^{m'}$, and $n=2^m3^{m'}5^{m''}$. Thus, $\{t_{k(3,5,7)}\}$, $\{t_{k(3,5,7,11)}\},\ldots$ are subsets of points t_k at which large $|Z(t)|$ are increasingly likely.

4.3. **Proximity of $\frac{\vartheta(t)}{\pi}$ to $\mathbb{N}$.** The Riemann-Siegel formula provides another hint about the location of large values of $|Z(t)|$. The mutually reinforcing terms (see Section 4.2) are proportional to $|\cos\vartheta(t)|$, which is the largest if t corresponds to a Gram point (a point $t=g_m>7$ such that $\vartheta(g_m)=m\pi$, $m\in\{-1,0\}\cup\mathbb{N}$). In fact, for each of the 105 ILE in $\mathcal{Z}_2\cup\mathcal{Z}_3$, either at the closest Gram point below t_k, or at the closest Gram point above t_k, $|Z(g_m)|$ exceeds 99.2% of the value at the local extremum.[2] Among the t_k that qualify both by proximity of $\frac{k\log p}{\log 2}$ to integers and by a large $|Z(t_k)|$, further selection of the candidates for ILE can thus be made by computing $|Z(t)|$ at the two Gram points closest to t_k.

[2]Of the two Gram points closest to t_k, it is *not always* the one closer to t_k at which $|Z(t)|$ is large (e.g., for $k=954$, the closest Gram point is $t=g_{8571}$, yet $|Z(t)|$ is larger at $t=g_{8570}$).

4.4. **Partial Riemann-Siegel sums at large $|Z(t)|$.** Let

$$ {}_rZ_0(t) := 2 \sum_{1\le n\le m} \frac{\cos(\vartheta(t) - t\log n)}{\sqrt{n}}, \quad \text{where } m = \left[\left(\frac{t}{2\pi} \right)^{1/(2+r)} \right], $$

so that $Z_0(t) \equiv {}_0Z_0(t)$. For 61 of the 62 ILE in $\mathcal{Z}_3$, the value of ${}_1Z_0$ at the corresponding point t_k exceeds 9.0% of the value of Z at the extremum. Furthermore, for all 62 ILE in $\mathcal{Z}_3$, the value of ${}_1Z_0$ (resp. ${}_2Z_0$) at one of the two Gram points closest to t_k exceeds 39.5% (resp. 19.6%) of the value of Z at the extremum. In all these cases, the sign of ${}_rZ_0$ at the considered point equals the sign of Z at the extremum. Thus, evaluation of ${}_1Z_0$ at points t_k and of either ${}_1Z_0$ or ${}_2Z_0$ at Gram points could be used for elimination of unlikely ILE candidates, significantly reducing the number of complete Z-evaluations.

5. Prospects for further progress

The analysis of the order of $Z(t)$ by means of the functions $a(t)$ and $b(t)$ is based on the rigorously established results, $Z(t) = O(t^A)$ and $Z(t) = \Omega(t^{B/\sqrt{\log t \, \log\log t}})$, and as such might be viewed as rather conservative. It would be tempting to evaluate a stronger Ω-conjecture than the one tested through $b(t)$, e.g., by considering the function $g(t) := \log|Z(t)|/\sqrt{\log t}$ to test the conjecture $Z(t) = \Omega(t^{G/\sqrt{\log t}})$ for some $G > 0$. However, the results of such a procedure could be misleading, as we have no knowledge of the multiplicative constant involved in the order of $Z(t)$. For example, the values of $|Z(t)|$ at ILE agree rather well (with the correlation coefficient $R = 0.9994$ for ILE with $t > 10^3$) with the estimate $|Z(t)| = 0.0199t^{3.36/\sqrt{\log t \, \log\log t}}$. If this were actually the case, then $g(t)$ at ILE would increase up to $t \approx 10^{89}$, and in any computationally accessible t-range one would be led to the wrong conclusion that $G > 0$. In other words, while $g(t)$ at ILE increases for $t \le 10^{13}$ and exceeds the value of 1, there is no guarantee that $\limsup_{t\to\infty} g(t) > 0$.

One might also be tempted to extrapolate. That is, if the functional forms of the asymptotes that the values of $a(t)$ and $b(t)$ at ILE seem to outline were identified correctly, say as $a_S(t)$ and $b_S(t)$, the respective limits as $t \to \infty$ would yield estimates of A and B. Yet, without any theoretical indications with respect to what the functions $a_S(t)$ and $b_S(t)$ should be, such an identification would amount to guessing, and it is unclear how one could assess its correctness. For example, the values of $a(t)$ at ILE agree reasonably well ($R = 0.9985$ for ILE with $t > 10^3$) with the power-decay function $a_S(t) = 0.149 + 0.255t^{-0.0528}$, which would suggest that $Z(t) = \Omega(t^{0.149})$. This estimate would contradict Lindelöf's (and hence Riemann's) hypothesis, and while it also agrees well with the data for $t < 10^{13}$, for sufficiently large t it is destined to run into a complete disagreement with the estimate of $|Z(t)|$ at ILE given in the previous paragraph.

It is sometimes supposed that if any violations of Riemann's hypothesis exist, they could be located close to very large values of $|Z(t)|$. There are no such violations in the vicinity of the 162 ILE determined in this study.

The computations presented in this paper took approximately nine months using a personal computer. At the time of writing, the most powerful supercomputers could have handled this task at least one thousand times faster. It is unlikely that a supercomputer would be dedicated somewhere to the search for further ILE, but this search could also be distributed among a number of personal computers, with the rate of advancement proportional to the total computing power of the computers involved.[3] In addition, the search could be accelerated by selecting ILE candidates through partial Riemann-Siegel sums (Section 4.4) and by computing the extrema using the Odlyzko-Schönhage algorithm [6].

Appendix

		t	$Z(t)$
$\overline{\mathcal{Z}}_1$	1	0.000000	–1.460
	2	10.212075	–1.552
	3	17.882582	2.341
	4	27.735883	2.847
	5	35.392730	2.942
	6	45.636113	–3.665
	7	63.060428	–4.167
	8	90.723857	4.477
	9	108.986791	5.193
	10	171.759106	–5.980
	11	199.651794	6.063
	12	245.532580	6.069
	13	280.810364	–7.003
	14	371.545466	7.570
	15	480.401432	–8.250
	16	652.212123	9.158
	17	897.836383	9.406
	18	1069.360643	9.851
	19	1178.449084	10.355
	20	1378.316536	–10.468
	21	1550.029928	11.077
	22	1967.268238	11.271
	23	2030.520469	11.730
	24	2447.635780	13.371
	25	3099.906368	13.479
	26	3825.816853	–13.497
	27	3997.707224	–13.575
	28	4478.096605	–14.755
	29	6726.121510	–15.612
	30	6925.621938	–15.955
	31	8475.812323	–16.252
	32	8647.210888	16.391
	33	9173.716528	16.506
	34	10025.578053	16.906
	35	10677.929307	–17.237
	36	11204.207758	17.337
	37	12645.135236	–18.006
	38	13125.470242	18.091
	39	14303.975890	19.817
	40	22299.074877	21.059
	41	24329.633861	21.434
	42	30774.966419	23.228
	43	50626.478383	23.747
	44	55104.583439	–24.830
	45	63751.863162	–26.073
	46	74956.025038	–27.694
	47	77403.722067	28.216
	48	105731.032300	28.853
	49	130060.556256	31.415
	50	152359.757336	32.671
	51	260538.282724	34.161
	52	314464.228643	34.516
	53	328768.228899	–36.689
	54	521928.541866	36.739
	55	534573.688201	–40.991
	56	865898.755362	–42.392
	57	929650.688269	–43.107
$\overline{\mathcal{Z}}_2$	58	1024177.378756	44.063
	59	1345367.802772	–47.593
	60	1923053.135018	–48.350
	61	2186410.518907	–50.879
	62	2939652.714358	53.233
	63	3268420.883436	–55.204
	64	3345824.546021	55.767
	65	5419578.489302	–58.425
	66	6155416.653707	61.038
	67	9850232.528074	–62.448
	68	9969615.203761	62.793
	69	11026769.624984	–65.674
	70	12372137.487612	–67.952
	71	15236834.026567	–68.116
	72	15457423.712975	74.268

[3]This strategy is being applied efficiently in an ongoing computation of the zeros of Riemann's zeta function, which has so far shown that Riemann's hypothesis holds for $|t| < 3 \times 10^{10}$ [9].

		t	$Z(t)$
	73	28642802.916415	–75.213
	74	28660206.960842	75.625
	75	30694257.761606	79.679
	76	37002034.097306	–80.035
	77	42792359.891727	–80.513
	78	46747714.116054	–82.469
	79	53325356.508449	–84.321
	80	60090302.842436	84.715
	81	81792403.155463	85.761
	82	82985411.177787	–86.254
	83	87568424.951600	91.882
	84	99273480.761352	–91.989
	85	102805259.027575	92.643
	86	119015924.891142	92.654
	87	124570459.059572	95.158
	88	144327207.118141	–95.326
	89	151614082.016804	97.031
	90	173723252.257957	–101.319
	91	178900422.227382	103.906
	92	244946055.644911	108.011
	93	298271412.198149	108.187
	94	363991205.176448	–114.451
	95	418878041.160027	–118.153
	96	607838127.431023	118.447
	97	631240860.404037	119.782
	98	673297382.192693	124.043
	99	868556070.995988	128.017
	100	900138526.590236	–128.993
$\overline{\mathcal{Z}_3}$	101	1189754916.313216	–130.488
	102	1253191043.688385	133.120
	103	1387123309.986048	148.728
	104	2287261836.552282	149.404
	105	3238682014.814266	149.611
	106	3443895116.936669	–152.488
	107	4209002696.395103	155.270
	108	4266153346.590529	157.986
	109	4945603697.701426	–160.578
	110	5230260126.511580	–164.581
	111	5272517912.850547	170.199
	112	7181324522.908048	–171.458
	113	7965404181.305970	–176.842
	114	11166740191.846172	180.227
	115	12251628740.237935	181.884
	116	13066290725.695175	183.530
	117	18168214001.673350	190.187
	118	19018488753.002784	192.635

	t	$Z(t)$
119	21559062801.941668	–192.996
120	22412382038.812786	–196.059
121	23165396411.338070	196.477
122	25985505104.438565	–197.606
123	27279224693.810314	204.462
124	27331684151.577735	209.054
125	31051083602.364182	213.898
126	38688523992.011831	224.263
127	62792807608.657779	–228.392
128	79881740253.040389	233.330
129	102108905446.095547	240.103
130	108903432915.370254	242.415
131	124855728535.680010	–246.885
132	131443859639.685072	251.267
133	133159989048.388546	251.576
134	165822762086.732367	–254.192
135	170165889140.424800	–256.095
136	192604855973.407448	258.354
137	197804421842.227818	–262.702
138	243860776768.360133	–271.338
139	297280771283.496679	–276.661
140	326473979757.428188	–289.781
141	461305748544.638105	292.784
142	472692195365.796730	–293.833
143	479489261691.339254	293.845
144	514119669706.650653	295.026
145	576555893019.852818	295.375
146	643049954739.247192	–297.567
147	669980906189.791285	301.088
148	722931694992.231828	309.299
149	812980259631.147353	–334.401
150	1459387308608.408274	349.779
151	1765497206246.212277	354.787
152	2515593134489.563683	–361.066
153	2589877332690.841810	370.395
154	3210707929490.468401	375.250
155	4154422573264.686997	–376.393
156	4778933265685.642359	379.550
157	5695465916337.181354	388.067
158	6586779209214.248987	–403.914
159	7709188977559.148583	405.312
160	8743721888758.038535	415.783
161	9090142088295.475463	416.329
162	9918400224732.229613	–441.106
	4257232978148261.797669	855.364

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References

[1] R. Balasubramanian and K. Ramachandra, *On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$*. III*, Proc. Ind. Acad. Sci. **86A** (1977), 341-351. MR **58:**21968

[2] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, 1974, pp. 176-177. MR **57:**5922

[3] M. N. Huxley, *Exponential sums and the Riemann zeta function. IV*, Proc. Lond. Math. Soc. **66** (1993), 1-40. MR **93j:**11056

[4] E. Lindelöf, *Quelques remarques sur la croissance de la fonction* $\zeta(s)$, Bull. Sci. Math. **32** (1908), 341-356.

[5] A. M. Odlyzko, *The* 10^{20}*-th zero of the Riemann zeta function and 175 million of its neighbors*, `http://www.dtc.umn.edu/~odlyzko/unpublished/index.html`

[6] A. M. Odlyzko and A. Schönhage, *Fast algorithms for multiple evaluations of the Riemann zeta function*, Trans. Am. Math. Soc. **309** (1988), 797-809. MR **89j:**11083

[7] J. B. Rosser, J. M. Yohe, and L. Schoenfeld, *Rigorous computation and the zeros of the Riemann zeta-function*, Proc. IFIP Congress 1968, North-Holland, 1969, pp. 70-76. MR **41:**2892

[8] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-function, 2nd ed.*, Oxford University Press, 1986, p. 354. MR **88c:**11049

[9] S. Wedeniwski, *ZetaGrid—Verification of the Riemann hypothesis*, `http://www.zetagrid.net/zeta/index.html`

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